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LETTER TO THE EDITOR

Generalized Hirota's bilinear equations and their soliton solutions

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Abstract. A generalized Hirota's bilinear equation is considered. Furthermore, two special forms of it are studied in some detail. Under certain conditions, we show that the equations have one- or two-soliton solutions. The conditions under which three-soliton or N-soliton solutions exist are also given. Some examples are illustrated.

Hirota's bilinear method is now an important part of soliton theory (see, e.g. [1,2]). The idea behind it is to use some transformation to put the nonlinear evolution equation in a form which is quadratic in the dependent variable(s) and where derivatives appear only through the bilinear operator defined below. It is known that many important nonlinear integrable equations such as KdV, Boussinesq and KP can be transformed into the following Hirota's bilinear equation

$$F(D_x, D_t, D_y) f \cdot f = 0 \tag{1}$$

where F is an even-degree polynomial and the bilinear operator $D_x^l D_t^m D_y^n$ is defined by

$$\begin{aligned} D_x^l D_t^m D_y^n a(x,t,y) \cdot b(x,t,y) \\ \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n \\ \times a(x,t,y) b(x',t',y')|_{x'=x,t'=t,y'=y}. \end{aligned}$$

Once a nonlinear evolution equation is transformed into a Hirota's bilinear equation, the latter seems to be easier to treat, e.g. to construct soliton solutions. Hirota proved [1] that there exist at least two-soliton solutions for (1) if $F(0,0,0) = 0$. Moreover a sufficient condition on the existence of N-soliton solutions for (1) was also given. Hietarinta [3] investigated the conditions for the existence of three-soliton solutions for (1) and found some new differential equations possessing three-soliton solutions. Unfortunately, equation (1) is only a very small class of nonlinear evolution equations. Therefore it is necessary to generalize (1) and then proceed to find more and more new integrable equations. In [4], we consider a generalized form of (1)

$$\sum_{k=1}^{l} H_k(D_x, D_t, D_y) \left[F_k(D_x, D_t, D_y) f \cdot f\right] \cdot \left[G_k(D_x, D_t, D_y) f \cdot f\right] = 0 \tag{2}$$

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where F_k, G_k, H_k are all polynomials of D_x, D_t, D_y with constant coefficients. By imposing certain conditions on F_k, G_k and H_k, we show that (2) possesses one-soliton solutions. A new integrable equation

$$D_x\left[(D_x^3D_t+\alpha_1D_t^2+\beta_1D_xD_t+\delta_1D_tD_y+\delta_2D_x^2)f\cdot f\right]\cdot f^2 + D_t\left[(\alpha_2D_xD_t^3+\alpha_2\delta_2D_t^2)f\cdot f\right]\cdot f^2=0 \tag{3}$$

with $\alpha_1,\alpha_2,\beta_1,\delta_1,\delta_2$ being arbitrary constants, is found which is an extension of the Novikov–Veselov equation and the Ito equation.

In this letter, we consider another generalization of (1)

$$\sum_{k=1}^{l}H_k(D_x,D_t)\left[F_k\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)\tau\right]\cdot\left[G_k\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)\tau\right]=0. \tag{4}$$

Here, for the sake of simplicity, we only consider the (1+1)-dimensional case and H_k, F_k, G_k are all polynomials with constant coefficients. If we set $H_k(0,0)=0, k=1,2,\ldots,l$, then it is easy to verify that (4) has the following one-soliton solution

$$\tau=1+e^{\eta}\qquad \eta=px+\Omega t+\eta^0$$

where p,Ω,η^0 are constants and

$$\sum_{k=1}^{l}\left[F_k(p,\Omega)G_k(0,0)H_k(p,\Omega)+F_k(0,0)G_k(p,\Omega)H_k(-p,-\Omega)\right]=0.$$

Now we consider two special form of (4).

First we consider the following special form of (4).

$$H(D_x,D_t)\left[F\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)\tau\right]\cdot\left[G\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)\tau\right]=0 \tag{5}$$

where F, G, H are all polynomials with constant coefficients, and

$$\begin{aligned}&H(0,0)=0 && |F(0,0)|^2+|G(0,0)|^2\neq 0\\ &H(-D_x,-D_t)=(-1)^{\epsilon}H(D_x,D_t) && \epsilon=0 \text{ or } 1.\end{aligned} \tag{6}$$

Then we have:

Proposition 1. Equation (5) with condition (6) has two-soliton solutions

$$\tau=1+e^{\eta_1}+e^{\eta_2}+A_{12}e^{\eta_1+\eta_2} \tag{7}$$

where $\eta_i=p_ix+\Omega_it+\eta_i^0$, $H(p_i,\Omega_i)=0$ for $(i=1,2)$, p_i, Ω_i, η_i^0 are constants and

$$\begin{aligned}A_{12}=&-[F(p_1,\Omega_1)G(p_2,\Omega_2)+(-1)^{\epsilon}F(p_2,\Omega_2)G(p_1,\Omega_1)]H(p_1-p_2,\Omega_1-\Omega_2)\\ &\times\left\{[G(0,0)F(p_1+p_2,\Omega_1+\Omega_2)\right.\\ &\left.+(-1)^{\epsilon}F(0,0)G(p_1+p_2,\Omega_1+\Omega_2)]H(p_1+p_2,\Omega_1+\Omega_2)\right\}^{-1}.\end{aligned}$$

Proof. Direct calculation.

Following Hirota's idea, we can show that equation (5) with condition (6) has N-soliton solutions

$$\tau = \sum_{\mu=0,1} \exp\left(\sum_{j=1}^{N} \mu_j \eta_j + \sum_{j>k}^{(N)} \mu_j \mu_k A_{jk} \right)$$

$$\eta_i = p_i x + \Omega_i t + \eta_i^0 \qquad H(p_i, \Omega_i) = 0 \qquad i = 1, 2, \ldots, N$$

with

$$\begin{aligned} e^{A_{jk}} = &-[F(p_j, \Omega_j)G(p_k, \Omega_k) + (-1)^\epsilon F(p_k, \Omega_k)G(p_j, \Omega_j)]H(p_j - p_k, \Omega_j - \Omega_k) \\ &\times \big\{[G(0,0)F(p_j + p_k, \Omega_j + \Omega_k) \\ &+ (-1)^\epsilon F(0,0)G(p_j + p_k, \Omega_j + \Omega_k)]H(p_j + p_k, \Omega_j + \Omega_k)\big\}^{-1} \end{aligned}$$

where $\sum_{\mu=0,1}$ indicates summation over all possible combinations of $\mu_1 = 0, 1, \mu_2 = 0, 1, \ldots, \mu_N = 0, 1$, $\sum_{j>k}^{(N)}$ means the summation over all possible combinations of N elements under the condition $j > k$, and p_i, Ω_i, η_i^0 $(i = 1, \ldots, N)$ are constants, provided that

$$\begin{aligned} \Delta_{mn} \equiv \sum_{\sigma=\pm 1} &F\left(\sum_{j=1}^{n} \tfrac{1}{2}(1+\sigma_j)p_j + \sum_{j=n+1}^{m} p_j,\ \sum_{j=1}^{n} \tfrac{1}{2}(1+\sigma_j)\Omega_j + \sum_{j=n+1}^{m} \Omega_j \right) \\ &\times G\left(\sum_{j=1}^{n} \tfrac{1}{2}(1-\sigma_j)p_j + \sum_{j=n+1}^{m} p_j,\ \sum_{j=1}^{n} \tfrac{1}{2}(1-\sigma_j)\Omega_j + \sum_{j=n+1}^{m} \Omega_j \right) \\ &\times H\left(\sum_{j=1}^{n} \sigma_j p_j, \sum_{j=1}^{n} \sigma_j \Omega_j \right) \\ &\times \Pi_{j>k}^{(n)} \Big[F\big(\tfrac{1}{2}(1-\sigma_j)p_j + \tfrac{1}{2}(1+\sigma_k)p_k, \tfrac{1}{2}(1-\sigma_j)\Omega_j + \tfrac{1}{2}(1+\sigma_k)\Omega_k\big) \\ &\times G\big(\tfrac{1}{2}(1-\sigma_k)p_k + \tfrac{1}{2}(1+\sigma_j)p_j, \tfrac{1}{2}(1-\sigma_k)\Omega_k + \tfrac{1}{2}(1+\sigma_j)\Omega_j\big) \\ &+ (-1)^\epsilon F\big(\tfrac{1}{2}(1-\sigma_k)p_k + \tfrac{1}{2}(1+\sigma_j)p_j, \tfrac{1}{2}(1-\sigma_k)\Omega_k + \tfrac{1}{2}(1+\sigma_j)\Omega_j\big) \\ &\times G\big(\tfrac{1}{2}(1-\sigma_j)p_j + \tfrac{1}{2}(1+\sigma_k)p_k, \tfrac{1}{2}(1-\sigma_j)\Omega_j + \tfrac{1}{2}(1+\sigma_k)\Omega_k\big) \Big] \\ &\times H(\sigma_k p_k - \sigma_j p_j, \sigma_k \Omega_k - \sigma_j \Omega_j)\sigma_j \sigma_k = 0 \end{aligned} \qquad (8)$$

for $1 < n < m < N + 1$ with

$$p_{N+1} = \Omega_{N+1} = 0.$$

In what follows, we give two examples.

Example 1.

$$(D_t - D_x)\tau_{tt} \cdot \tau = 0. \tag{9}$$

In this case, $F(\partial/\partial x, \partial/\partial t) = \partial^2/\partial t^2$, $G(\partial/\partial x, \partial/\partial t) = 1$, $H(D_x, D_t) = D_t - D_x$. It is easily verified that (8) is satisfied. So (9) has N-soliton solutions. Note that

$$H(p_j - p_k, \Omega_j - \Omega_k) = H(p_j + p_k, \Omega_j + \Omega_k) = 0$$

thus A_{jk} can be chosen arbitrarily. Now we introduce $u = (\ln\tau)_t$, $v = \frac{1}{2}(u_t - u_x)$, then (9) can be transformed into

$$u_t = u_x + 2v \qquad v_t = -2uv$$

which is just the Tu equation [5].

Example 2.

$$(D_t - D_x^3)\left[\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\tau\right] \cdot \tau = 0. \tag{10}$$

Equation (10) also belongs to (5). According to proposition 1, (10) has two-soliton solutions. Introduce $u = (\ln\tau)_x$, $v = (\partial/\partial t - \partial^2/\partial x^2)\tau/\tau$, then (10) can be transformed into

$$u_t = u_{xx} + 2uu_x + v_x \qquad v_t = v_{xxx} + 6u_x v_x.$$

Next we consider another special form of (4)

$$H(D_x, D_t)F(D_x, D_t)\left[F\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)\tau\right] \cdot \left[G\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)\tau\right] = 0 \tag{11}$$

with

$$H(0,0)F(0,0) = 0. \tag{12}$$

We have the following result:

Proposition 2. Equation (11) with condition (12) has two-soliton solutions

$$\tau = e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2} \tag{13}$$

with

$$\eta_i = p_i x + \Omega_i t + \eta_i^0 \qquad F(p_i, \Omega_i) = 0 \qquad (i = 1, 2)$$

where p_i, Ω_i, η_i^0 for $(i = 1, 2)$ are constants, and A_{12} is an arbitrary constant.

Proof. Direct calculation.

Further we assume

$$F(-D_x, -D_t) = (-1)^{\epsilon} F(D_x, D_t) \qquad \epsilon = 0 \text{ or } 1 \tag{14}$$

and set

$$\tilde{H}(D_x, D_t) = H(D_x, D_t)F(D_x, D_t).$$

Then we have:

Proposition 3. Equation (11), with conditions (12) and (14), has three-soliton solutions

$$\tau = \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_3} + A_{12}\mathrm{e}^{\eta_1+\eta_2} + A_{13}\mathrm{e}^{\eta_1+\eta_3} + A_{23}\mathrm{e}^{\eta_2+\eta_3} + A_{123}\mathrm{e}^{\eta_1+\eta_2+\eta_3} \tag{15}$$

with $\eta_i = p_i x + \Omega_i t + \eta_i^0$, $F(p_i, \Omega_i) = 0$ with $(i = 1, 2, 3)$ and p_i, Ω_i, η_i^0 $(i = 1, 2, 3)$, $A_{12}, A_{13}, A_{23}, A_{123}$ being constants, provided that

$$\begin{aligned} &A_{12}F(p_1 + p_2, \Omega_1 + \Omega_2)G(p_3, \Omega_3)\tilde{H}(p_1 + p_2 - p_3, \Omega_1 + \Omega_2 - \Omega_3) \\ &\quad + A_{13}F(p_1 + p_3, \Omega_1 + \Omega_3)G(p_2, \Omega_2)\tilde{H}(p_1 + p_3 - p_2, \Omega_1 + \Omega_3 - \Omega_2) \\ &\quad + A_{23}F(p_2 + p_3, \Omega_2 + \Omega_3)G(p_1, \Omega_1)\tilde{H}(p_2 + p_3 - p_1, \Omega_2 + \Omega_3 - \Omega_1) = 0 \end{aligned} \tag{16}$$

$$\begin{aligned} &A_{12}A_{13}[F(p_1 + p_2, \Omega_1 + \Omega_2)G(p_1 + p_3, \Omega_1 + \Omega_3)\tilde{H}(p_2 - p_3, \Omega_2 - \Omega_3) \\ &\quad + F(p_1 + p_3, \Omega_1 + \Omega_3)G(p_1 + p_2, \Omega_1 + \Omega_2)\tilde{H}(p_3 - p_2, \Omega_3 - \Omega_2)] \\ &\quad + A_{123}F(p_1 + p_2 + p_3, \Omega_1 + \Omega_2 + \Omega_3)G(p_1, \Omega_1)\tilde{H}(p_2 + p_3, \Omega_2 + \Omega_3) = 0 \end{aligned} \tag{17}$$

$$\begin{aligned} &A_{12}A_{23}[F(p_1 + p_2, \Omega_1 + \Omega_2)G(p_2 + p_3, \Omega_2 + \Omega_3)\tilde{H}(p_1 - p_3, \Omega_1 - \Omega_3) \\ &\quad + F(p_2 + p_3, \Omega_2 + \Omega_3)G(p_1 + p_2, \Omega_1 + \Omega_2)\tilde{H}(p_3 - p_1, \Omega_3 - \Omega_1)] \\ &\quad + A_{123}F(p_1 + p_2 + p_3, \Omega_1 + \Omega_2 + \Omega_3)G(p_2, \Omega_2)\tilde{H}(p_1 + p_3, \Omega_1 + \Omega_3) = 0 \end{aligned} \tag{18}$$

$$\begin{aligned} &A_{13}A_{23}[F(p_1 + p_3, \Omega_1 + \Omega_3)G(p_2 + p_3, \Omega_2 + \Omega_3)\tilde{H}(p_1 - p_2, \Omega_1 - \Omega_2) \\ &\quad + F(p_2 + p_3, \Omega_2 + \Omega_3)G(p_1 + p_3, \Omega_1 + \Omega_3)\tilde{H}(p_2 - p_1, \Omega_2 - \Omega_1)] \\ &\quad + A_{123}F(p_1 + p_2 + p_3, \Omega_1 + \Omega_2 + \Omega_3)G(p_3, \Omega_3)\tilde{H}(p_1 + p_2, \Omega_1 + \Omega_2) = 0. \end{aligned} \tag{19}$$

Proof. Direct calculation.

In the following, we give some examples.

Example 3.

$$(D_t - D_x^2)\left[\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\tau\right]\cdot\tau = 0. \tag{20}$$

This equation belongs to (11) with condition (12), so (20) has two-soliton solutions. Introduce $u = (\ln\tau)_x$, $v = -\frac{1}{2}(\partial/\partial t - \partial^2/\partial x^2)\tau/\tau$, then (20) can be transformed into

$$u_t = u_{xx} + 2uu_x - 2v_x \qquad v_t = v_{xx} + 2u_x v$$

which is the first equation of the two-truncated KP hierarchy [6].

Example 4.

$$(D_t - D_x^3)\left[\left(\frac{\partial}{\partial t} - \frac{\partial^3}{\partial x^3}\right)\tau\right]\cdot\tau = 0. \tag{21}$$

This equation belongs to (11) with conditions (12) and (13). In this case, $H(D_x, D_t) = 1$, $F(D_x, D_t) = D_t - D_x^3$, $G(\partial/\partial x, \partial/\partial t) = 1$. By some calculations, we can show that (16)–(19) are satisfied and

$$A_{12} = \frac{(p_1+p_3)(p_2+p_3)}{(p_1-p_3)(p_2-p_3)}(p_1-p_2)\sqrt{\frac{p_3 A_{123}}{p_1 p_2(p_1+p_2+p_3)}}$$

$$A_{13} = \frac{(p_1+p_2)(p_2+p_3)}{(p_1-p_2)(p_1-p_3)}(p_1-p_2)\sqrt{\frac{p_2 A_{123}}{p_1 p_3(p_1+p_2+p_3)}}$$

$$A_{23} = \frac{(p_1+p_2)(p_1+p_3)}{(p_1-p_2)(p_1-p_3)}(p_2-p_3)\sqrt{\frac{p_1 A_{123}}{p_2 p_3(p_1+p_2+p_3)}}.$$

So (21) has a three-soliton solution

$$\tau = e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3} + A_{123}e^{\eta_1+\eta_2+\eta_3}$$

where $\eta_i = p_i x + \Omega_i t + \eta_i^0$, $\Omega_i - p_i^3 = 0$ for $(i = 1, 2, 3)$. Introduce $u = (\ln\tau)_x$, $v = (\partial/\partial t - \partial^3/\partial x^3)\tau/\tau$, then (21) can be transformed into

$$u_t = u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u^2u_x + v_x \qquad v_t = v_{xxx} + 6u_x v_x. \tag{22}$$

Example 5.

$$(D_t^2 - D_x^2)\left[\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\tau\right]\cdot\tau = 0. \tag{23}$$

Equation (23) belongs to (11) with $H(D_x, D_t) = 1$, $F(D_x, D_t) = D_t^2 - D_x^2$, $G(\partial/\partial x, \partial/\partial t) = 1$. By some calculations, we can show that (16)–(19) are satisfied and

$$A_{12} = \sqrt{\frac{A_{123}(J_{12}+J_{13}+J_{23})(J_{13}+J_{23})}{(J_{12}+J_{13})(J_{12}+J_{23})}}$$

$$A_{13} = \sqrt{\frac{A_{123}(J_{12}+J_{13}+J_{23})(J_{12}+J_{23})}{(J_{12}+J_{13})(J_{13}+J_{23})}}$$

$$A_{23} = \sqrt{\frac{A_{123}(J_{12}+J_{13}+J_{23})(J_{12}+J_{13})}{(J_{12}+J_{23})(J_{13}+J_{23})}}$$

with

$$J_{ij} = \Omega_i\Omega_j - p_i p_j.$$

So (23) has a three-soliton solution

$$\tau = e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3} + A_{123}e^{\eta_1+\eta_2+\eta_3}$$

according to proposition 3, where

$$\eta_i = p_i x + \Omega_i t + \eta_i^0 \qquad \Omega_i^2 - p_i^2 = 0 \qquad (i = 1, 2, 3).$$

Introduce $u = (\ln\tau)_x$, $v = (\partial^2/\partial t^2 - \partial^2/\partial x^2)\tau/\tau$, then (23) can be transformed into

$$u_{tt} + u_t^2 = u_{xx} + u_x^2 + v \qquad v_{tt} + 2vu_{tt} = v_{xx} + 2vu_{xx}. \tag{24}$$

Finally, we give some concluding remarks. First of all, the results in this letter can be generalized to higher-dimensional cases. Secondly by means of computer algebraic languages such as MACSYMA, MAPLE, REDUCE and MATHEMATICA, it should be possible to find more new differential equations having three-soliton solutions. For example, we guess that the equation

$$(D_t - D_x^5)\left[\left(\frac{\partial}{\partial t} - \frac{\partial^5}{\partial x^5}\right)\tau\right]\cdot\tau = 0 \tag{25}$$

has three-soliton solutions. It is a tedious task to check this by hand. However, it is practical to check whether or not (25) has three-soliton solutions by computer algebraic languages.

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